On the global warping of a thin self–gravitating near Keplerian gaseous disk with application to the disk in NGC 4258

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To appear in ApJ

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ABSTRACT

We derive the tilt equation governing the inclination of a thin self–gravitating gaseous disk subject to low frequency global $m = 1$ bending perturbations. The disk orbits under the influence of a dominant central mass. However, self–gravity can be important enough that the disk approaches marginal stability to local axisymmetric perturbations ($Q \sim 1$). The vertical restoring forces due to self–gravity and pressure are evaluated correct to first order in the aspect ratio H/r. Thus the effects of bending waves are included correct to lowest order in the wave crossing rate $(H/r)\Omega$, Ω being a characteristic disk rotation frequency.

Both free and forced disturbances are considered. The disk response and precession frequency induced by the presence of a binary companion in an orbit with general inclination to the unperturbed disk plane are derived. When the degree of warping and the inclination are small, it is shown that identical results are obtained if, alternatively, perturbation of the disk out of an equilibrium plane coinciding with that of the companion is considered, the time averaged potential due to the latter being incorporated into the equilibrium potential. The condition for the disk to precess approximately like a rigid body with a small degree of warping is found to be that the density wave crossing time be significantly shorter than the precession period.

We consider the effects of the presence of a viscosity which can be characterized with the standard α parameterization and find that, to the order we work, the precession frequency is unaffected, with any change to the inclination of the slightly warped disk occuring at a slower rate. For $\alpha \ll H/r$, effects due to both pressure and self–gravity are important in the response, while for $\alpha \gg H/r$, the response becomes dominated by self–gravity with pressure effects becoming negligible.

As an application of these results, we explore the possibility that the recently observed warped disk in the active galaxy NGC 4258 is produced by a binary companion. Our results indicate that it may be produced by a companion with a mass comparable to or larger than that of the observed disk. If the only source of warping is such a companion, a small trailing twist is produced by viscosity for modest $\alpha \sim 0.1$. The outer edge of the disk may also be tidally truncated by the companion.

Subject headings: accretion, accretion disks — galaxies: kinematics and dynamics — galaxies: structure — galaxies: individual (NGC 4258)

1. Introduction

Gaseous disk configurations which are dominated by a central mass and in which internal self–gravity is important may occur in different astrophysical contexts such as protostellar disks (Larson 1984) or active galactic nuclei (Paczynski 1977).

The most definitive evidence for a gaseous accretion disk in active galactic nuclei is provided by the discovery of megamasers (Claussen, Heiligman & Lo 1984) around the nucleus of the mildly active galaxy NGC 4258 (Claussen & Lo 1986). Confirmation of Keplerian rotation was obtained from the radio interferometer (Nakai, Inoue & Miyoshi 1993) and VLBI observations (Greenhill *et al.* 1995). Based on the correlation between the spatial locations and radial velocities of the masers, Miyoshi *et al.* (1995) deduced that the masers are located at $R \sim 0.13 - 0.26$ pc around a black hole with a mass 3.6×10^7 $M_\odot$ (Watson & Wallin 1994; Maoz 1995). Recently the inner part of the disk has been modelled with an advection–dominated flow (Lasota *et al.* 1996).

Random deviations from the Keplerian rotation curve at the location of the masers are ~ 3.5 km/s. The lack of systematic deviations provides an upper limit for the disk mass of 4×10^6 $M_\odot$. The disk scale height $H < 3 \times 10^{-4}$ pc is such that $H/R < 2.5 \times 10^{-3}$ and the mid–plane temperature $T_c < 10^3$ K (Moran *et al.* 1995). In addition, the high–velocity maser sources have negligible acceleration and are not colinear with the low–velocity masers (Greenhill *et al.* 1995). The radial dependence of declination of the red–shifted (with respect to the systemic velocity of NGC 4258) high–velocity masers is antisymmetric to that of the blue–shifted high–velocity masers (Miyoshi *et al.* 1995). One scenario to account for these observed properties is that the position angle of the rotation axis varies continually with radius by up to 0.2 radians (Herrnstein, Greenhill & Moran 1996). This corresponds to introducing a small warp into the disk model. The inclination of the local orbital plane with respect to the orbital plane at the innermost region of the disk, g, being the ratio of the vertical displacement to the local radius, varies with the disk radius. In at least one set of models, small–amplitude variations in the angle between the line of sight and the rotation axis are also introduced. This corresponds to introducing a “twist” into the warped disk. The magnitude and direction of the twist are not yet well constrained.

In this paper, we study the dynamics of the warped disk in NGC 4258. The main constraints for any theoretical model are: 1) the coherence and the smooth radial dependence in the inclination g, 2) the negligible acceleration in the high velocity maser sources, and 3) the nearly perfect Keplerian rotation curve.

We explore the possibility that the observed warp may be produced by a binary companion which orbits in a plane inclined to that of the disk. Such non coplanar system could be produced if the companion approaches the disk as a result of random gravitational perturbations arising from the field stars in the nucleus of NGC 4258 (see § 4). Indeed, in that case there is no *a priori* reason to suppose alignment between its orbital angular momentum vector and that of the matter which supplies the disk at large distances. Furthermore, in the evolutionary situation we envisage, the disk is not expected to have settled to an equilibrium in which its mid–plane coincides with

the companion's orbital plane (see § 1.3). Capture of a companion is not expected to result systematically in the disruption of the disk. The orbit may have indeed become bound when the companion was far away from the disk, with a subsequent decrease of the separation due to dynamical friction.

In the model we shall adopt in this paper, the disk is subject to forces due to internal self–gravity, pressure, viscosity, and also to the gravitational forces arising from the central object and the perturbing companion on an inclined orbit. Since we shall focus on the secular response of the disk (see § 2.2), the perturbation is the same as that produced by a mass distributed uniformly along an inclined ring. For a near Keplerian self–gravitating disk with small viscosity, pressure may in principle be as important as self–gravity in determining the response. However, for the model we adopt in § 3 for NGC 4258, self–gravity turns out to dominate over pressure and viscosity in determining the disk response to the perturbing potential. Before going on to develop the tilt equation we use to describe near Keplerian disks, we summarize some general results on the dynamics of warped disks.

1.1. Precession of warped disks

In a classic paper, Hunter & Toomre (1969) showed that an isolated self–gravitating disk subject to a vertical displacement generally precesses differentially, and thus cannot sustain a warped configuration. However, differential precession is prevented by gravitational torques from the distorted disk itself if the disk has a sharp edge. A clear physical description of how self–gravity acts to smooth out differential precession is given by Toomre (1983). A similar process occurs if the disk orbits in the external potential due to a companion (Hunter & Toomre 1969) or a flattened halo whose equatorial plane is misaligned with the disk plane (Toomre 1983; Dekel & Shlosman 1983; Sparke 1984; Sparke & Casertano 1988), provided departure from spherical symmetry of the external potential is relatively small. In this case, the disk settles into a discrete bending mode (representing a warp) which is referred to as the modified tilt mode because in the limit that the external potential is spherically symmetric it reduces to the trivial rigid tilt mode. The difference in shape and frequency between the rigid and modified tilt modes is due to the fact that, in a non spherically symmetric potential, the disk has to bend to alter the precession frequency at each radius so that the rate is everywhere the same (Sparke & Casertano 1988; Hofner & Sparke 1994). We note that the potential of an oblate halo can be modeled by that of a distant massive fixed ring which is misalinged with the disk plane (Toomre 1983). This situation is analogous to that being investigated here in this paper, namely a near Keplerian self–gravitating disk dominated by a central mass and a smaller orbiting companion which in a time average sense gives the appearance of a massive ring.

Papaloizou & Terquem (1995) have shown that radial pressure forces in a non self–gravitating inviscid accretion disk are also able to smooth out differential precession. This process can be effective if the sound crossing time through the disk is much smaller than the precession

period. When this condition is satisfied, bending waves are able to propagate through the disk sufficiently fast so that the different parts of the disk can "communicate" with each other and adjust their precession rate to a constant value. This also happens when viscosity is present, but the communication becomes diffusive rather than wave–like when the Shakura & Sunyaev (1973) viscosity parameter α significantly exceeds the ratio of disk semi–thickness to radius (see Papaloizou & Pringle 1983; Demianski & Ivanov 1997). The theoretical expectation of uniform precession of inclined disks under conditions of adequate physical communication has been confirmed by the numerical simulations of Larwood *et al.* (1996).

When the disk precesses uniformly in an external potential, its precession frequency can be calculated from the condition that the disk is stationary in a rotating frame. This condition implies that the net torque exerted by the external potential and the Coriolis force is zero (by virtue of Newton's third law, the disk does not exert a torque on itself). Kuijken (1991) has used this condition to calculate the precession frequency of a self–gravitating disk in a flattened halo whose plane of symmetry is misaligned with the disk plane. In the limit of small misalignment, his result reduces to that given by perturbation theory (Sparke & Casertano 1988). As expected, his results also indicate that when the departure from spherical symmetry of the potential gets too large, self–gravity can no longer maintain a uniform precession of the entire disk. In the context of simulations of gaseous disks in interacting binary stars that were misaligned with the binary orbital plane, Larwood *et al.* (1996) found that when the tidal potential of the companion is large enough, forces due to pressure and viscosity in a non self–gravitating thin disk cannot prevent differential precession.

1.2. Timescale for settling to the warp mode

We have pointed out above that the disk settles into a (modified tilt) mode which enables it to undergo rigid precession. This asymptotic state is possible because bending waves transport away the energy associated with the transient response (Toomre 1983). In a purely self–gravitating disk, this energy is carried out towards the disk outer edge by bending waves whose group velocity approaches zero there. The waves pile up with very short wavelength at the edge where they would be expected to dissipate. The timescale for settling to the warp mode is given by the characteristic time for density waves to propagate through the disk but the process is slowed down in the low surface density regions near the outer edge. Hofner & Sparke (1994) found that when a galactic disk is subject to the potential of a misaligned flattened halo, the inner parts settle first and the outer parts are unlikely to settle into the warp mode within a Hubble time.

When pressure is present the waves may be reflected from the edge before they attain arbitrary short wavelength there. Some dissipation is then needed for the disk to settle into a state of near rigid precession. This can be provided by disk viscosity which is least effective on a global disturbance such as near rigid body precession. Other transient disturbances would, as we have indicated above, have shorter wavelengths and accordingly would be expected to dissipate

significantly more rapidly. In this context note that a warp mode which deviates only slightly from a rigid tilt has a wavelength which is much larger than the disk. Larwood *et al.* (1996) considered the disk response due to the perturbation of a companion. In their simulations, of non self–gravitating inclined disks with pressure and viscosity, the disk was seen to quickly settle into a state of near rigid body precession. The time for this to happen was consistent with about $\sim 1/\alpha$ orbital periods, this being the decay time of short wavelength bending waves without self–gravity (see Papaloizou & Lin 1994, 1995).

In this paper, we consider the limiting case of a very thin self–gravitating disk (see § 3.1) in which bending waves propagate with a speed comparable to that of sound and the settling timescale into the warp mode is short compared to the viscous evolution timescale of the entire disk.

1.3. Timescale for settling to the orbital plane

Papaloizou & Terquem (1995) have shown that if a non self–gravitating disk perturbed by a companion on an inclined orbit evolves towards the symmetry plane of the potential (which does not necessarily occur), it does so on a timescale related to the rate at which angular momentum is transported to the companion which is expected to be at least the disk viscous timescale. This conclusion might be expected since the disk inclination as a whole can change only if angular momentum is transferred between different parts of the disk and then between the disk and the companion's orbit. As the long term disk evolution is driven through the action of viscous stress, adjustment should occur so that angular momentum transfer to the orbit, and hence changes to disk inclination occur on that timescale or longer. Such an adjustment can occur for example through evolution of the location of the disk outer edge away from the companion's orbit.

This was seen in the numerical simulations of Larwood *et al.* (1996) and Larwood (1997), in which disk inclination was ultimately seen to evolve much more slowly than the precession rate. We also show that this is expected to occur for a mildly warped self–gravitating disk in section 2.4.6 below. Thus we do not necessarily expect the inclination of the disk to have evolved so that it coincides with the orbital plane.

Results of numerical simulations of the dynamical evolution of rings highly inclined to the symmetry plane of some galactic potential (Katz & Rix 1992; Christodoulou *et al.* 1992; Christodoulou & Tohline 1993) are also in agreement with the above conclusions. However, we emphasize that they only hold if the disk is able to find a state in which differential precession can be controlled by internal self–gravity and pressure. Settling toward a preferred orientation is much faster if differential precession is so important that it cannot be controlled by other forces (Steiman–Cameron & Durisen 1988). This happens, for example, when the external potential is such that the precession frequency is always not very small compared to the orbital angular frequency. However, this is not the case in the near Keplerian disks we consider. There the precession frequency is relatively very small at all inclinations.

1.4. Plan of the paper

The plan of the paper is as follows. The response of the disk to the companion perturbing potential is analyzed in § 2. We first give the basic equations and the expression for the time averaged perturbing potential we later use. We have taken the latter to be due to a companion on an inclined orbit, but any $m = 1$ perturbing potential proportional to the coordinate z along the disk rotation axis could equally well be considered. We then describe the disk equilibrium structure, linearize the basic equations and derive the tilt equation governing global variations in the disk inclination taking into account self–gravity and pressure. All effects contributing to low frequency bending wave propagation across the disk are included to leading order.

The disk response and precession frequency are derived using two different approaches, which are shown to be equivalent in the limit of small inclination of the companion orbit. We consider i) the excitation of small amplitude low–frequency bending modes, considered as perturbations of the disk away from the companion orbital plane. Here the time averaged potential of the companion is included in the equilibrium state. We also consider ii) the zero frequency response of a disk forced by the time averaged potential due to a companion in a general inclined orbit. Here the companion is not included in the equilibrium state. We go on to discuss the effect of viscosity, showing how this has the effect of reducing the importance of the pressure forces in comparison to those due to self–gravity once the viscosity parameter α significantly exceeds H/r. We also show that when the disk warping is small, the disk inclination evolves on a long timescale.

In § 3 we apply these results to NGC 4258. We derive the disk parameters from the observations and present the numerical results. We find that, if the disk is self–gravitating, the observed warp may be produced by a companion of comparable mass to that of the disk, with an orbital separation of about one and a half times the observed outer radius of the disk. The presence of the companion may also cause the disk truncation. We also show that viscous forces induce a twist in the orientation of the spin axes of annuli at different radii. For the estimated disk parameters, only a small trailing twist is expected to be associated with the warp if a perturbing companion alone produces it. At the present moment, there is no strong constraints on the magnitude of the twist. But an accurate measure of it can provide useful information on the non conservative forces acting in the disk.

We summarize and discuss our results in § 4.

2. Disk response

2.1. Basic equations

The dynamics of the disk is described by the equation of motion

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\,\mathbf{v} = -\frac{1}{\rho}\boldsymbol{\nabla} P - \boldsymbol{\nabla}\Psi + \mathbf{f}_{\mathbf{v}}, \tag{1}$$

and the equation of continuity

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0, \tag{2}$$

where P is the pressure, ρ the density, $\mathbf{v}$ the flow velocity and Ψ the total gravitational potential. We allow for the possible presence of a viscous force per unit mass $\mathbf{f}_{\mathbf{v}}$ but we shall assume that it does not affect the undistorted axisymmetric disk so that it operates on the perturbed flow only.

We write $\Psi = \Psi_{ext} + \Psi_G$, where Ψ_{ext} is the contribution to potential due to the central black hole and the perturber and Ψ_G is the contribution to the potential arising from the disk self–gravity given by

$$\Psi_G = -G \int_V \frac{\rho\,(\mathbf{r}')\,d^3\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}, \tag{3}$$

the integral being taken over the disk volume, with $\mathbf{r}$ and $\mathbf{r}'$ denoting position vectors and G being the gravitational constant.

For our calculations, we adopt a polytropic equation of state $P = K\rho^{1+1/n}$, K and n being the polytropic constant and index respectively. Then Ω is a function of r alone, and the sound speed is given by $c_s^2 = dP/d\rho$.

2.2. Perturbing potential

We consider the response of the thin self–gravitating disk to a perturbing gravitational potential that has a plane of symmetry which is inclined to that of the disk. We suppose that the disk is perturbed by a point mass M_p orbiting as a binary companion to the point mass M located at the disk centre. In our presentation, the disk outer radius is denoted by R and the binary separation by D.

We consider a non rotating Cartesian coordinate system (x, y, z) centered on the central point mass M. The z axis is chosen to be along the rotation axis of the unperturbed disk. We shall also use the associated cylindrical polar coordinates (r, φ, z). We take the orbit of the perturbing mass to be in a plane which has an inclination angle δ with respect to the (x, y) plane and the line of nodes to coincide with the x axis. We denote the position vector of the perturbing mass by $\mathbf{D}$ with $D \equiv |\mathbf{D}|$.

The potential Ψ_{ext} arising from both the central and the orbiting masses is given by

$$\Psi_{ext} = -\frac{GM}{\mid \mathbf{r} \mid} - \frac{GM_p}{\mid \mathbf{r} - \mathbf{D} \mid} + \frac{GM_p \mathbf{r}.\mathbf{D}}{D^3}$$

The last indirect term accounts for the acceleration of the origin of the coordinate system. We are interested in the warping of the disk which is excited by terms in the potential which are odd in z and which have azimuthal mode number $m = 1$ when a Fourier analysis in φ is carried out. We consider only the secular term in the potential, that is to say the zero–frequency term, because this is the dominant contribution for producing the large–scale warp structure, in which we are interested. This term is obtained by taking a time average of the total perturbing potential, and is equivalent to replacing the full potential by the one obtained if the perturbing mass is distributed uniformly along the orbit. Non zero–frequency terms would be important in determining the disk response if it extended much beyond the inner Lindblad (2:1) resonance with the companion. Here we suppose that truncation of the disk, through the action of bending and other waves excited by interaction with the companion, occurs such that $D = 1.5R$ (see for example Lin & Papaloizou 1993), which more or less excludes the resonances required for effective wave excitation from the disk. Non secular bending waves propagating into the disk interior have short wavelength and do not give a global response. This is supported by the numerical simulations of Larwood *et al.* (1996) who found that the global precession of truncated disks could be accounted for by the secular response.

The term in the Fourier expansion of the potential which is of the required form is given by

$$\Psi'_{ext} = \frac{\sin\varphi}{4\pi^2} \int_0^{2\pi} d(\omega t) \int_0^{2\pi} \left[\Psi_{ext}(r, \varphi', z, \omega t) - \Psi_{ext}(r, \varphi', -z, \omega t) \right] \sin\varphi' d\varphi', \tag{4}$$

where ω is the companion's orbital frequency. The parameters we shall use in the numerical calculations require a development to eighth order in D^{-1} of this integral:

$$\Psi'_{ext} = -\frac{3}{4} \frac{GM_p}{A^{5/4} D^3} \, rz \; \sin 2\delta \; \sin\varphi \left[1 + \frac{a_1}{A} \left(\frac{r}{D} \right)^2 + \frac{a_2}{A^2} \left(\frac{r}{D} \right)^4 + \frac{a_3}{A^3} \left(\frac{r}{D} \right)^6 + \frac{a_4}{A^4} \left(\frac{r}{D} \right)^8 \right], \tag{5}$$

with $A = \left(1 + r^2/D^2\right)^2$ and

$$\begin{aligned}
a_1 &= 2.1875 \left(2 - 1.5 \sin^2 \delta \right), \\
a_2 &= 6.0156 \left(3 - 4.5 \sin^2 \delta + 1.875 \sin^4 \delta \right), \\
a_3 &= 18.3289 \left(4 - 9 \sin^2 \delta + 7.5 \sin^4 \delta - 2.1875 \sin^6 \delta \right), \\
a_4 &= 59.2022 \left(5 - 15 \sin^2 \delta + 18.75 \sin^4 \delta - 10.9375 \sin^6 \delta + 2.4609 \sin^8 \delta \right).
\end{aligned}$$

Since we consider the thin disk limit, additional terms, smaller by factors containing powers of z/r, have been neglected. Throughout this paper we shall use the complex potential

$$\Psi'_{ext} = i\,\frac{3}{4}\frac{GM_p}{A^{5/4}D^3}\,rf(r)z\ \sin 2\delta\ e^{i\varphi}, \tag{6}$$

defined such that its real part is equal to the physical potential. The function $f(r)$ is the term in brackets in (5).

2.3. Equilibrium structure of the disk

In the absence of perturbation, the disk is axisymmetric so that in cylindrical coordinates the velocity $\mathbf{v} = (0, r\Omega, 0)$ and the hydrostatic equilibrium equations are satisfied in the form

$$\frac{1}{\rho}\frac{\partial P}{\partial r} = -\frac{\partial \Psi}{\partial r} + r\Omega^2, \tag{7}$$

$$\frac{1}{\rho}\frac{\partial P}{\partial z} = -\frac{\partial \Psi}{\partial z}. \tag{8}$$

In the absence of perturbation or orbiting companion, the disk is under the influence of the central point mass M so that $\Psi_{ext} = -GM/\sqrt{r^2+z^2}$. In the present context, we consider thin disks for which the radial scale over which physical parameters vary is very much larger than the vertical scale height. In the limit where the Toomre parameter Q is of order unity, the disk's self–gravity, which may be evaluated as if the disk had zero thickness, is then more important by a factor of order r/H than pressure in (7). The angular velocity is given by equation (7) as

$$\Omega^2 = \Omega_K^2 + \frac{1}{r\rho}\frac{\partial P}{\partial r} + \frac{1}{r}\frac{\partial \Psi_G}{\partial r}, \tag{9}$$

where $\Omega_K = \left(GM/r^3\right)^{1/2}$ is the Keplerian angular velocity. Typically the contribution of self–gravity gives $\Omega - \Omega_K = O(H\Omega_K r^{-1})$. As a check, we evaluated the contribution of the pressure term for the models we consider here and verified that it was relatively small.

We could also add the time average potential due to a companion with small mass, which orbits in the disk plane, into Ψ_{ext} and incorporate it in the equilibrium. This would give a contribution of order ω_z, being the orbital precession frequency of a free disk particle, to $\Omega - \Omega_K$. However, we shall limit ourselves to companion masses small enough so that $|\omega_z| \ll (H\Omega r^{-1})$ and its effect may in general be considered to be small in comparison to that due to self–gravity. But, in a formal sense, we shall consider that typically $|\omega_z| \gg (H^2\Omega r^{-2})$, so that it may be retained while frequencies of order $H^2\Omega r^{-2}$ may be neglected.

The warping and precession of a disk at small inclination may be considered as a small amplitude perturbation about such a state incorporating a companion (e.g. Toomre 1983; Sparke 1984) and see below.

In principle, the surface density distribution, $\Sigma = \int_{-\infty}^{\infty} \rho dz$, is determined by the viscous evolution of the disk. In the absence of a deterministic prescription for the effective viscosity, for illustrative purposes we arbitrarily choose $\Sigma(r) = \Sigma_0 \left(R/r - 1 \right)$, Σ_0 being a constant. Alternative functional forms describing the way Σ tapers to zero have also been considered below.

2.4. Linear perturbations

In the limit where either any perturbing/forcing potential is small compared to that due to the central mass and the amplitude of any free bending modes is small, the disk response can be described in a linear analysis. We here follow the standard procedure of vertical averaging (see, for example, Hunter & Toomre 1969, Sparke 1984, Papaloizou & Lin 1995). This should be valid when the radial wavelength of the disturbance is significantly longer than the disk thickness as is the case for the perturbations considered here. For linear warps, the dependence on φ and t may be taken to be through a factor $\exp[i(m\varphi + \sigma t)]$. Henceforth this factor will be taken as read. For the time being we shall leave it as a multiplicative factor for all perturbations but will remove it at a later stage. The mode frequency, σ, can be taken to be zero for a secular response viewed in an appropriate rotating reference frame and we shall consider this case below. Alternatively, the disturbance may appear with a small non zero frequency or pattern speed when viewed in an inertial frame. In the latter case, the disk will appear to precess if the perturbation has a global form. We limit ourselves to considering frequencies significantly less than the inverse density wave crossing time of the disk (characteristically $O(H\Omega r^{-1})$) which is a necessary condition if the disk is to exhibit global precession (Hofner & Sparke 1994, Papaloizou & Terquem 1995). Then we show below that, for modest warping of the disk, these two descriptions are equivalent. Finally, we remark that the azimuthal mode number $m = 1$ throughout.

We denote the Lagrangian displacement as $\boldsymbol{\xi} \equiv (\xi_r, \xi_\varphi, \xi_z)$. Here we shall not assume at the outset that the horizontal components of the displacement are zero, as it was shown in Papaloizou & Lin (1995) that these can produce significant effects in a self–gravitating but near Keplerian disk of the type considered here.

The inclination (vertical displacement/radius) associated with the tilt of the disk is ξ_z/r. This is in general assumed to be a slowly varying function of both r and z such that its variation with z may be neglected. The applicability of this vertical averaging approximation is supported by the numerical calculations of Papaloizou & Lin (1995) which allowed for the possibility of variation of the vertical displacement associated with modes of the type we consider with z.

We first assume that we are working in an inertial frame in which the unperturbed disk appears steady. However, it may subsequently be convenient to adopt a uniformly rotating frame so as to

remove the rigid body precession associated with the motion of the disk, in which case the effects due to the coriolis force need to be added.

We begin by writing down the perturbed form of the vertical component of the equation of motion applicable to a gaseous disk with a barotropic equation of state (see Hunter & Toomre 1969)

$$\frac{D^2\xi_z}{Dt^2} = -\frac{\partial \Psi'_G}{\partial z} - \frac{\partial\left(P'/\rho\right)}{\partial z} - \frac{\partial \Psi'_{ext}}{\partial z}, \tag{10}$$

where perturbations to quantities are denoted with a prime and the convective time derivative D/Dt is here equivalent to multiplication by $i(\sigma + \Omega)$.

The potential perturbation, Ψ'_{ext}, is taken to be the secular contribution due to the companion in an inclined circular orbit (see § 2.2) where this has not been included in the equilibrium. Then we calculate the disk response taking $\sigma = 0$, but will need to transform to a rotating frame (see below). If the companion orbits in the initial symmetry plane of the disk and its effect included in the equilibrium, then we look for free bending modes with small frequency $O(\omega_z)$, the amplitude of which will give the inclination of the precessing disk. These two approaches are equivalent for small inclinations and modest warping.

For a polytropic equation of state we also have $P' = \rho' c_s^2$. Using the above, equation (10) may be written

$$(\sigma + \Omega)^2 \xi_z = \frac{\partial \Psi'_G}{\partial z} + \frac{\partial\left(\rho' c_s^2/\rho\right)}{\partial z} + \frac{\partial \Psi'_{ext}}{\partial z}. \tag{11}$$

The perturbation to the gravitational potential of the disk is given by the Poisson integral

$$\Psi'_G = -G \int_V \frac{\rho'\left(\mathbf{r}'\right) d^3\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}, \tag{12}$$

while the perturbed continuity equation gives

$$\rho' = -\boldsymbol{\nabla} \cdot \left(\rho \boldsymbol{\xi}\right). \tag{13}$$

We then find after an integration by parts, assuming that the disk density vanishes at its boundaries, that

$$\Psi'_G = -G \int_V \frac{\rho\left(\mathbf{r}'\right) \boldsymbol{\xi}\left(\mathbf{r}'\right) \cdot \left(\mathbf{r} - \mathbf{r}'\right) d^3\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}. \tag{14}$$

In what follows, we find it convenient to consider the contributions to ρ' and Ψ'_G arising from the horizontal and vertical components of $\boldsymbol{\xi}$ separately. The contributions arising from the vertical

component we denote with a subscript v and call them vertical contributions. The contributions arising from the horizontal components we denote with a subscript h and call them horizontal contributions. Thus

$$\rho'_v = -\frac{\partial(\rho\xi_z)}{\partial z}, \tag{15}$$
$$\rho'_h = -\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\xi}_h). \tag{16}$$

Ψ'_{Gv} and Ψ'_{Gh} denote the potential arising from ρ'_v, and ρ'_h respectively. Thus

$$\Psi'_{Gv} = -G\int_V \frac{\rho(\mathbf{r}')\,\xi_z(\mathbf{r}')\,(z-z')\,d^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}, \tag{17}$$
$$\Psi'_{Gh} = -G\int_V \frac{\rho(\mathbf{r}')\,\boldsymbol{\xi}_h(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')\,d^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{18}$$

The vertical and horizontal contributions give rise to additive contributions, F'_v and F'_h respectively, to the vertical component of the equation of motion (11). We consider these in turn below.

2.4.1. The tilt equation

Having evaluated the vertical and horizontal contributions we may construct a tilt equation governing the inclination g derived from vertical integration of equation (11) in the form

$$(\sigma+\Omega)^2\,\Sigma\,\xi_z = \int_{-\infty}^{\infty}\rho\,(F'_h+F'_v)\,dz + \int_{-\infty}^{\infty}\rho\,\frac{\partial\Psi'_{ext}}{\partial z}\,dz. \tag{19}$$

2.4.2. The vertical contribution

From equation (11), this is seen to give rise to the vertical deceleration

$$F'_v = \frac{\partial\Psi'_{Gv}}{\partial z} + \frac{\partial\left(\rho'_v c_s^2/\rho\right)}{\partial z}. \tag{20}$$

To deal with this, we differentiate the vertical contribution to the Poisson integral (17) with respect to z, differentiate the Poisson integral (3) twice with respect to z and combine with ξ_z to obtain in the limit of an arbitrary thin disk in its midplane

$$\frac{\partial\Psi'_{Gv}}{\partial z} + \xi_z\frac{\partial^2\Psi_G}{\partial z^2} \to -G\int_A \frac{\Sigma(\mathbf{r}')\,[\xi_z(\mathbf{r}')-\xi_z(\mathbf{r})]\,d^2\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\ , \tag{21}$$

where the two dimensional integral is over the surface of the disk.

Proceeding further, we remark here that we shall eventually require

$$\int_{-\infty}^{\infty} \rho F_v' dz = -\int_{-\infty}^{\infty} \rho \left(\xi_z \frac{\partial^2 \Psi_G}{\partial z^2} + G \int_A \frac{\Sigma(\mathbf{r}') \left[\xi_z(\mathbf{r}') - \xi_z(\mathbf{r}) \right] d^2\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right) dz - \int_{-\infty}^{\infty} \rho \frac{\partial}{\partial z} \left(\frac{c_s^2}{\rho} \frac{\partial(\rho \xi_z)}{\partial z} \right) dz. \tag{22}$$

Here we have made use of (21) and (15). Equation (22) may be further simplified by using the assumption that ξ_z is independent of z, together with use of the fact that the equilibrium quantities satisfy hydrostatic equilibrium (8) in the vertical direction, with the result that

$$\int_{-\infty}^{\infty} \rho F_v' dz = \Sigma\, \xi_z \frac{\partial^2 \Psi_{ext}}{\partial z^2} - G \int_A \frac{\Sigma(\mathbf{r}) \Sigma(\mathbf{r}') \left[\xi_z(\mathbf{r}') - \xi_z(\mathbf{r}) \right] d^2\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \ . \tag{23}$$

Here and below, we make the assumption that the second derivatives of the external potential are slowly varying through the disk thickness so they can be evaluated in the midplane. This completes the reduction of the vertical contribution to the force per unit area which now depends only on the vertical displacement.

2.4.3. The horizontal contribution

From equation (11), this is seen to give rise to the horizontal deceleration

$$F_h' = \frac{\partial \Psi_{Gh}'}{\partial z} + \frac{\partial \left(\rho_h' c_s^2 / \rho \right)}{\partial z}. \tag{24}$$

Noting again that we need to perform a vertical average, we multiply by ρ and integrate over the disk thickness to obtain after an integration by parts of the second term and use of vertical hydrostatic equilibrium

$$\int_{-\infty}^{\infty} \rho F_h' dz = \int_{-\infty}^{\infty} \left(\rho \frac{\partial \Psi_{Gh}'}{\partial z} + \rho_h' \frac{\partial \Psi_G}{\partial z} \right) dz + \int_{-\infty}^{\infty} \rho_h' \frac{\partial \Psi_{ext}}{\partial z} dz. \tag{25}$$

We pause for a moment to consider the magnitudes of the terms in equation (25). Referring back firstly to equation (23), we comment that the first term on the right hand side there is of order $\Omega_K^2 \Sigma \xi_z$ while the second, being produced by the self–gravity of the disk, is characteristically smaller by a factor H/r for Toomre parameter Q of order unity and global perturbations as considered here. Allowing for the possibility that the horizontal displacement could become as large in magnitude as the vertical one, the first two terms in equation (25), being produced by the disk self–gravity, are each also of potential magnitude $\Omega_K^2 \Sigma H \xi_z / r$. However, we show in the

Appendix that, for global perturbations, the combination is in fact smaller by an additional factor of order H/r, and so can be neglected in our ordering scheme. Thus only the final term remains in equation (25). This is best dealt with by considering the perturbed equations of motion in Eulerian form. These can be written (remembering $m = 1$)

$$\begin{aligned} i\left(\sigma+\Omega\right)v_r' - 2\Omega v_\varphi' &= -\frac{\partial W}{\partial r}, & (26)\\ i\left(\sigma+\Omega\right)v_\varphi' + \frac{\kappa^2}{2\Omega}v_r' &= -\frac{iW}{r}, & (27)\\ i\left(\sigma+\Omega\right)v_z' &= -\frac{\partial W}{\partial z}. & (28) \end{aligned}$$

Here the velocity perturbation is $\mathbf{v}' = (v_r', v_\varphi', v_z')$, $W = P'/\rho + \Psi'$, and $\kappa^2 = 2\Omega r^{-1} d\left(r^2\Omega\right)/dr$ is the square of the epicyclic frequency.

Given that we seek an inclination $\xi_z/r \equiv v_z'/\left[ir\left(\sigma+\Omega\right)\right]$ that is independent of z, we may write $W = \left(\sigma+\Omega\right)^2 z\xi_z$. Equations (26) and (27) may then be used to solve for the horizontal velocity perturbations in terms of ξ_z. These can then be used to find ρ_h' from the perturbed continuity equation (13). An analagous procedure has been carried out in Papaloizou & Lin (1995) and Papaloizou & Terquem (1995). As we are here interested in the low frequency limit, it is consistent with our ordering scheme to perform this procedure adopting $\sigma = 0$. We thus obtain

$$\begin{aligned} v_r' &= \frac{-i\Omega z}{r^2\left(\kappa^2-\Omega^2\right)}\frac{\partial\left(\xi_z r^2\Omega^2\right)}{\partial r}, & (29)\\ v_\varphi' &= \frac{\kappa^2 z}{2r^2\Omega\left(\kappa^2-\Omega^2\right)}\frac{\partial\left(\xi_z r^2\Omega^2\right)}{\partial r} - \frac{z\xi_z\Omega}{r}. & (30) \end{aligned}$$

Finally we obtain from (13)

$$\rho_h' = -\frac{1}{i\Omega}\boldsymbol{\nabla}\cdot\left(\rho\mathbf{v}_h'\right) = \frac{z}{\Omega r^{1/2}}\frac{\partial}{\partial r}\left(\frac{\rho\Omega}{r^{3/2}\left(\kappa^2-\Omega^2\right)}\frac{\partial\left(\xi_z r^2\Omega^2\right)}{\partial r}\right). \qquad (31)$$

In forming the above, consistently with the neglect of frequencies of order $H^2\Omega_K r^{-2}$, we have neglected the contribution of the last term in (30) to (31).

Using equation (25), we find the horizontal contribution to the vertically integrated vertical component of the equation of motion

$$\int_{-\infty}^{\infty}\rho F_h' dz = \frac{\partial^2\Psi_{ext}}{\partial z^2}\frac{1}{\Omega r^{1/2}}\frac{\partial}{\partial r}\left(\frac{\mu\Omega}{r^{3/2}\left(\kappa^2-\Omega^2\right)}\frac{\partial\left(\xi_z r^2\Omega^2\right)}{\partial r}\right), \qquad (32)$$

where we have approximated $\partial\Psi_{ext}/\partial z = (\partial^2\Psi_{ext}/\partial z^2)z$, the second derivative being evaluated on the midplane, and $\mu = \int_{-\infty}^{\infty} \rho z^2 dz$.

2.4.4. Reduction of the tilt equation

Having evaluated the vertical and horizontal contributions, we construct the tilt equation (19) which now appears as an equation for ξ_z alone.

Using equations (23) and (32), we obtain

$$\left[(\sigma+\Omega)^2 - \frac{\partial^2\Psi_{ext}}{\partial z^2}\right]\Sigma\,\xi_z + G\int_A \frac{\Sigma\left(\mathbf{r}\right)\Sigma\left(\mathbf{r}'\right)\left[\xi_z\left(\mathbf{r}'\right)-\xi_z\left(\mathbf{r}\right)\right]d^2\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3}$$

$$= r\,\Omega_K^2\,\frac{\partial}{\partial r}\left(\frac{\mu\Omega_K^2}{\kappa^2-\Omega^2}\frac{\partial\left(\xi_z/r\right)}{\partial r}\right) + \Sigma\,\frac{\partial\Psi'_{ext}}{\partial z}. \tag{33}$$

Here, consistently with the neglect of frequencies of order $H^2\Omega_K r^{-2}$, we have made the replacements $\partial^2\Psi_{ext}/\partial z^2 = \Omega^2 = \Omega_K^2$ in (32) everywhere apart from in the denominator $\kappa^2 - \Omega^2$. Also in the last term, the derivative is evaluated on the midplane.

Rather than retaining ξ_z as the quantity to be determined, we adopt $g = i\left(\xi_z/r\right)\exp\left(-i\varphi\right).$ At zero frequency, its modulus is the ratio of vertical to azimuthal velocity and also the disk inclination. Remembering that the azimuthal dependence of ξ_z is through a factor $\exp\left(i\varphi\right),$ the replacement with g which is accordingly independent of φ will remove this factor from equation (33). Carrying out this reduction we obtain

$$\left[(\sigma+\Omega)^2 - \frac{1}{r}\frac{\partial\Psi_G}{\partial r} - \frac{\partial^2\Psi_{ext}}{\partial z^2}\right]\frac{\Sigma g}{\Omega_K^2} = \mathcal{L}\left(g\right) + \frac{i\Sigma}{r\Omega_K^2}\frac{\partial\Psi'_{ext}}{\partial z}, \tag{34}$$

where the operator $\mathcal{L}$ is defined through

$$\mathcal{L}\left(g\right) = \frac{G}{r^3\Omega_K^2}\int_{R_i}^{R}\hat{K}\left(r,r'\right)\Sigma(r)\Sigma(r')\left[g(r)-g(r')\right]r'^2r^2dr' + \frac{\partial}{\partial r}\left(\frac{\mu\Omega_K^2}{\kappa^2-\Omega^2}\frac{\partial g}{\partial r}\right). \tag{35}$$

Here the kernel $\hat{K}$ is given by

$$\hat{K}(r,r') = \int_0^{2\pi}\frac{\cos\Phi d\Phi}{\left[r^2+r'^2-2rr'\cos\Phi\right]^{3/2}}, \tag{36}$$

and R_i is the innermost disk radius.

We also note that after using (7), for the low frequencies we consider, to the accuracy we are working

$$\Omega^2 - \frac{1}{r}\frac{\partial \Psi_G}{\partial r} - \frac{\partial^2 \Psi_{ext}}{\partial z^2} = \frac{1}{r}\frac{\partial \Psi_{ext}}{\partial r} - \frac{\partial^2 \Psi_{ext}}{\partial z^2} = 2\Omega_K \omega_z,$$

where again $\omega_z(r)$ is the orbital precession frequency due to the companion if it orbits in the symmetry plane and is included in the equilibrium external potential so that $\Psi'_{ext} = 0$. If instead the effect of a companion in an inclined orbit is included only through Ψ'_{ext}, we have $\omega_z = 0$.

Neglecting σ^2, the inviscid tilt equation is then written in the convenient form:

$$\frac{2\,(\sigma + \omega_z)\,\Sigma g}{\Omega_K} = \mathcal{L}(g) + \frac{i\Sigma}{r\Omega_K^2}\frac{\partial \Psi'_{ext}}{\partial z}. \quad (37)$$

We comment that the above equation describes the global response of a thin near Keplerian self–gravitating disk with Toomre parameter Q of order unity. The largest frequencies neglected are of order $H^2\Omega_k r^{-2}$. It can also be used to describe low frequency bending modes when the forcing term is set to zero. There are two identifiable contributions to the operator $\mathcal{L}$ defined by (35). The first comes from self–gravity and leads to the description of warps given by Hunter & Toomre (1969), Sparke (1984), Sparke & Casertano (1988), Kuijken (1991) and others. The second term in $\mathcal{L}$ can be identified as arising from pressure and it can lead to comparable effects to those due to self–gravity in a near Keplerian disk, in contrast to a galactic disk, because of the near Lindblad resonance that arises in the former case because $\kappa - \Omega = O(H\Omega r^{-1})$. The form of this term is the zero frequency global counterpart of that occurring in the low frequency WKB dispersion relations given by Papaloizou & Lin (1995) and Masset & Tagger (1996).

Another useful property noted by Sparke & Casertano (1988) in the pure self–gravity problem is that, for a finite isolated disk, $\mathcal{L}$ is self–adjoint, that is for any pair (g_1, g_2) that satisfy appropriate regularity conditions, including zero derivative at a Lindblad resonance if the pressure term is included:

$$\int_{R_i}^{R} g_2^* \mathcal{L}\,(g_1)\,dr = \left(\int_{R_i}^{R} g_1^* \mathcal{L}\,(g_2)\,dr\right)^*. \quad (38)$$

Furthermore, when $\omega_z = 0$ and there is no forcing, equation (37) has the solution $g =$ constant which coresponds to a rigid tilt in a pure Keplerian potential. Modification of this rigid tilt mode by a companion results in a low frequency modified tilt mode that describes global warping and precession of the disk.

2.4.5. Disk precession

i) Low frequency bending modes

We first consider the case of a disk precessing at low inclination about the angular momentum vector associated with the companion orbit. This can be viewed as the excitation of a small amplitude bending mode with the time averaged potential of the companion being included in the equilibrium state. Implicit in this treatment is the notion that the companion orbit is fixed so its angular momentum content must be presumed to be significantly larger than that of the disk. Such modes are governed by the eigenvalue problem

$$\frac{2\,(\sigma+\omega_z)\,\Sigma g}{\Omega_K} = \mathcal{L}(g). \tag{39}$$

When $\omega_z = 0$, there is a solution corresponding to the rigid tilt mode with $g = g_0 = \text{constant} = \delta$, where for convenience we take δ to be real and therefore equal to the inclination and $\sigma = 0$. For small ω_z, there is a solution with g close to g_0 and σ non zero but small in magnitude. From first order perturbation theory, remembering $\mathcal{L}(g_0) = 0$, we obtain the forced response problem for g:

$$\mathcal{L}(g) = \frac{2\,(\sigma+\omega_z)\,\Sigma\delta}{\Omega_K}. \tag{40}$$

Multiplying by g_0, integrating over the disk and using the self–adjoint property of $\mathcal{L}$, we find the integrability condition which determines σ:

$$\sigma\int_{R_i}^{R}\frac{\Sigma g_0\delta}{\Omega_K}dr = -\int_{R_i}^{R}\frac{\Sigma\omega_z g_0\delta}{\Omega_K}dr. \tag{41}$$

Remembering that g_0 is constant and the form of Kepler's law, this may be written

$$\sigma = -\int_{R_i}^{R} 2\pi r\Sigma(r)\omega_z(r)j(r)\delta dr \bigg/ \int_{R_i}^{R} 2\pi r\Sigma(r)j(r)\delta dr\ , \tag{42}$$

where $j(r) = r^2\Omega_K$ denotes the specific angular momentum of the orbiting disk material.

This relation has a simple interpretation, as the gyroscope equation, that the magnitude of the disk precession frequency $-\sigma$ should equal the component of the applied torque perpendicular to the companion angular momentum axis and the disk angular momentum axis divided by the magnitude of the component of the disk angular momentum vector perpendicular to the companion angular momentum axis (Kuijken 1991). This relation is applicable to secular precession of free particle orbits and thus determines ω_z, so $\omega_z j(r)\delta = -T(r)$, where $T(r)$ is the external torque component per unit mass acting on a disk circular orbit at radius r.

Once the precession frequency has been found, the forced response problem can be solved to find the degree of warping. For the idea of global precession to be valid this should be small. This condition is roughly equivalent to the requirement that the disk wave crossing time be short compared to the precession period.

Another usefull relation that can be obtained from (39) is found by multiplying by g^* and integrating over the disk, giving

$$2\sigma \int \frac{\Sigma g^* g}{\Omega_K} dr + 2 \int \frac{\omega_z \Sigma g^* g}{\Omega_K} dr = \int g^* \mathcal{L}(g) dr. \tag{43}$$

The eigenfrequency σ, if real, corresponds to a simple precession. However, if it were complex, the imaginary part would give the rate of decay of the inclination towards the symmetry plane. From (43) we obtain for this decay rate

$$Im(\sigma) = Im \left(\int g^* \mathcal{L}(g) dr \right) \Big/ \left(2 \int \frac{\Sigma g^* g}{\Omega_K} dr \right). \tag{44}$$

Because of the self–adjoint property of the problem considered up to now, the decay rate is zero. However, if contributions from viscous or other non conservative forces are added, a non zero decay rate may result (see below).

ii) External forcing at finite inclination

In this case, we consider the zero frequency response of a disk forced by the secular potential due to a companion in an inclined orbit given by (6) so that $\omega_z = \sigma = 0$, and the governing equation is

$$\mathcal{L}(g) = -\frac{i\Sigma}{r\Omega_K^2} \frac{\partial \Psi'_{ext}}{\partial z}. \tag{45}$$

But because the unforced problem has $g =$ constant as a solution, the forced problem will not in general have a solution. Physically, this is because there is an unbalanced torque due to the companion which produces disk precession. To deal with this, we suppose the disk appears steady in a frame precessing about the total angular momentum axis with frequency ω_p. At first, let us suppose this coincides with the companion orbital angular momentum vector as in the preceeding section. Then we must add the vertical component of the coriolis force as a perturbing force along with that due to the external potential. This amounts to an additional perturbing force per unit mass being the real part of $-2i\omega_p \sin(\delta) r\Omega_K \exp{(i\varphi)}$. Incorporating this into equation (40) (i.e. adding it to $-\partial \Psi'_{ext}/\partial z$) results in this now becoming

$$\mathcal{L}(g) = \frac{\Sigma}{r\Omega_K^2} \left(2\omega_p r\Omega_K \sin\delta - i\frac{\partial \Psi'_{ext}}{\partial z} \right). \tag{46}$$

The above problem is very similar to, and in fact equivalent to, (40) for small δ. Just as in that case, we have a forced response problem to solve for g and we find the integrability condition that determines ω_p to be

$$\omega_p = \int_{R_i}^{R} \frac{i\Sigma}{2r\Omega_K^2} \left(\frac{\partial \Psi'_{ext}}{\partial z} \right) dr \Bigg/ \int_{R_i}^{R} \frac{\Sigma \sin \delta}{\Omega_K} dr. \tag{47}$$

This equation can also be interpreted as a gyroscope equation giving the precesion frequency as the ratio of appropriate disk torque and angular momentum components. Note here that the perturbing external potential is complex (see 6) ensuring that the calculated precession frequency is real and that the factor of two in (47) accounts for an azimuthal average.

After finding ω_p from (47), equation (46) may be integrated to give g, to within the addition of an arbitrary constant inclination, which may be eliminated by choosing the coordinate system so that $g = 0$ at the disk inner boundary. Then if g is small elsewhere, the disk approximately precesses like a rigid body.

We have mentioned in § 1.3 that the disk is not expected to have settled to equilibrium with zero inclination to the orbital plane. The choice of zero inclination perturbation at the inner boundary is then justified.

From (5), we see that $\omega_p \propto \cos \delta$ when $D \gg R$ such that the tidal potential becomes quadrupolar. The same dependance is found for a disk subject to the potential of a misaligned flattened halo, δ being in that case the angle between the disk plane and the symmetry plane of the potential (Kuijken 1991).

Up to now we have assumed that the disk angular momentum content can be neglected in comparison to that of the orbiting companion. Then the disk precesses about the companion angular momentum axis. For finite disk angular momentum content, the disk and companion both precess about the conserved total angular momentum vector. This can be taken account of by noting that the precession frequency ω_p as calculated above should be replaced by $\omega_p J_{orb}/J$, where J_{orb} is the companion's orbital angular momentum and J is the total angular momentum.

2.4.6. The effect of viscosity

In principle viscosity can act on the disk so as to change its inclination with respect to the orbital plane (see § 1.3), and also by damping the horizontal velocities that can be amplified by a close Lindblad resonance (see equations 29 and 30).

We have already observed in § 1.3 that, as long as it is globally warped, the timescale on which the disk inclination is expected to change is the global viscous timescale, namely $\Omega^{-1}(R/H)^2/\alpha$, where α is the standard Shakura & Sunyaev (1973) parameter. Since $\alpha \leq 1$, evolution on this

timescale is long and has been neglected in comparison to the other processes considered in the above analysis, where frequencies on the order of $H^2\Omega r^{-2}$ have been neglected. Thus, in the ordering scheme we have adopted, changes in the disk inclination with respect to the orbital plane should be negligible.

In a near Keplerian disk, viscosity also provides a damping effect on the near resonantly produced horizontal velocity perturbations. We follow the discussion given by Papaloizou & Lin (1994, 1995). The near resonant denominator acting in equations (29) and (30) is $1 - \kappa^2/\Omega^2 \sim O(H/r)$ when $Q \sim 1$. The effect of an appropriately defined α viscosity, at the order we are working, is to replace this denominator by $(1 - i\alpha)^2 - \kappa^2/\Omega^2$. Demianski & Ivanov (1997) have recently given the relativistic generalization. Thus viscosity can only be neglected when $\alpha \ll H/r$, which is not the case in some models of NGC 4258 (see below).

As indicated above, we take viscosity into account by changing the denominator $\kappa^2 - \Omega^2$ of the last term of the operator $\mathcal{L}$ (equation 35) to $\kappa^2 - \Omega^2(1 - i\alpha)^2$. We note that when $\alpha \gg H/r$, the pressure term in $\mathcal{L}$ is small compared to that due to self–gravity, so that the warp is mainly controlled by self–gravity. This is illustrated below in the case of models for NGC 4258.

We comment that incorporation of viscosity as described above does not affect the discussion leading to the determination of the precession frequency through equation (47). This is consistent with the idea that as long as the disk is only mildly warped, the internal viscous forces do not enable a net torque to be exerted on the disk, and so do not modify the precession rate or change the disk inclination except possibly on a long timescale.

In the case of small inclination we may use (44) to find the rate of decay of the inclination after incorporating the effect of viscosity as outlined above. For this we then obtain

$$Im(\sigma) = \int \frac{2\alpha\mu\Omega^2\Omega_K^2}{\left[\kappa^2 - \Omega^2\left(1-\alpha^2\right)\right]^2 + 4\alpha^2\Omega^4} \left|\frac{\partial g}{\partial r}\right|^2 dr \Bigg/ \left(2\int \frac{\Sigma g^* g}{\Omega_K} dr\right). \tag{48}$$

At a given location in the disk, the decay rate is a maximum for $\alpha \sim H/r$. For $\alpha > H/r$, the decay rate is found to be of order $\Omega(H/r)^2\alpha\left[\delta g/(\alpha g)\right]^2$, where δg represents the total change in inclination or the total degree of warping in the system. Thus the decay rate is small if the total warp is small.

3. Application to NGC 4258

A warped disk model was originally proposed for NGC 4258 to account for the origin of the megamasers (Neufield & Maloney 1995). In this scenario, the warp provides a favorable condition for parts of the disk to be illuminated by the X–ray source within the nucleus. In the regions between $0.13 - 0.26$ pc, the surface layers of the disk are heated to temperatures in the range $300\ K < T < 8,000\ K$ at which water maser production may occur (Collison & Watson 1995).

Neufield & Maloney (1995) supposed that conditions are less favourable for reprocessing in those regions of the disk interior to 0.13 *pc* and exterior to 0.26 *pc*.

With respect to the systemic velocity of NGC 4258, the flux of the red–shifted masers is an order of magnitude larger than that of the blue–shifted masers. Herrnstein *et al.* (1996) suggested that the heated gas above the concave side of a warped disk leads to greater extinction than that above the convex side. With the appropriate orientation, thermal absorption is dominant for the blue–shifted masers.

3.1. Disk Parameters

For the total luminosity of $L = 4 \times 10^{41}$ erg/s attributed to this object, an efficiency ϵ (here taken to be 0.1) of rest mass conversion implies a mass accretion rate through the system of

$$\dot{M} = \frac{L}{\epsilon c^2} = 6 \times 10^{-5}\ M_\odot/yr. \tag{49}$$

Note that the need for reprocessing is evident as according to a steady state optically thick disk model (see Pringle 1981) the effective temperature would be given by

$$T_{eff} = 11(r_{18})^{-0.75}\ K,$$

where r_{18} is the radius r in units of 10^{18} cm. For comparison, the effective temperature, assuming that the central luminosity is radiated over a radial scale r_{18}, would be

$$T_{eff} = 154(r_{18})^{-0.5}\ K.$$

Thus, in the most region of the disk, the irradiation is the dominant source of heating and T_{eff} is in the range appropriate for water maser emission.

The disk semi–thickness H is given by

$$\frac{H}{r} \sim \frac{c_s}{r\Omega} \sim 1.5 \times 10^{-3} \left(\frac{T_{200} r_{18}}{M_{7.6}}\right)^{1/2}. \tag{50}$$

Here $M_{7.6} = M/(4 \times 10^7\ M_\odot)$, M is the central black hole's mass, c_s is the sound speed, $T_{200} = T_c/200\ K$, T_c is the midplane temperature, and Ω is the orbital rotational frequency in the disk. From the observed upper limit $H/R < 2.5 \times 10^{-3}$, we find $T_c \sim T_{eff}$ due to irradiation and $T_{200} r_{18}/M_{7.6} < 2.8$.

Applying the conventional α prescription for an effective turbulent viscosity (Shakura & Sunyaev 1973), the mass transfer rate in a steady–state disk is

$$\dot{M} = 3\pi\alpha H^2\Omega\Sigma = 0.59\alpha\frac{M_D}{M}\left(\frac{T_{200}^2 M_{7.6}}{r_{18}}\right)^{1/2} \; M_\odot/yr, \tag{51}$$

where we have replaced $\pi\Sigma r^2$, with Σ being the surface density, by M_D, the disk mass interior to radius r. For NGC 4258, we find from (51) and (49) that

$$\alpha \sim 10^{-4}\frac{M}{M_D}\left(\frac{T_{200}^2 M_{7.6}}{r_{18}}\right)^{-1/2}. \tag{52}$$

The importance of self–gravity is measured by the Toomre Q parameter, $Q \sim (HM)/(rM_D)$, such that $Q \geq 1$ is required for stability to axisymmetric ring modes. From (50) we find that the disk becomes gravitationally unstable in this way if $M_D/M \geq 1.5 \times 10^{-3}$. From (52) we find $\alpha \sim 0.1$ for $M_D/M = 10^{-3}$. A smaller value of α would imply larger Σ, small Q value and a more unstable disk.

Based on the above estimates, we consider a model which is marginally self–gravitating with $Q \sim 1$ and $\alpha \sim 0.1$ such that $M_D \sim 3 \times 10^4 \; M_\odot$. We note that Maoz (1995) has suggested that the radial intervals between successive maser sources may be related to the wavelength of transient spiral structure in a marginally self–gravitating disk. The value of α we have assumed is comparable to that expected to occur in a marginally self gravitating disk as a result of non–axisymmetric waves (Laughlin & Rozyczka 1996).

The inner part of the disk of NGC 4258 has been modelled by Lasota *et al.* (1996) as being advection–dominated. In this case there would be greater mass flow rate through the disk than indicated by the observed luminosity. In principle this situation could be obtained by taking a somewhat thicker more massive disk together with a larger value of α. As indicated below, our results may be scaled to apply to such a case.

Here, we consider the response of a self–gravitating disk, with parameters like those discussed above, to an orbiting binary companion in an inclined circular orbit. This interaction produces warping and precession of the disk. We note that angular momentum transfer between the companion and disk may produce truncation and gap formation in the disk (Lin & Papaloizou 1993), terminating the radial progression of maser sources.

Pringle (1996) has indicated that forces due to radiation pressure may be important in producing warps in disks. Such forces could be included in the formalism presented here. However, we shall limit ourselves in this application to discussing the effects arising from a companion only.

In our model, the companion then needs to have a mass comparable to that of the disk and an orbital radius somewhat larger than the size of the disk. Black hole binary systems in which the companions can have up to comparable mass and separations on a scale of 10^{18} *cm* have been considered by Begelman, Blandford & Rees (1980), and postulated to account for jet behavior

in extragalactic sources by Kaastra & Roos (1992) and Roos, Kaastra & Hummel (1993). The dense cores of globular clusters may be an alternative candidate for such a companion. From the negative intrinsic period derivative of millisecond pulsars, Phinney (1992) derived a mass $> 4.5 \times 10^3\ M_\odot$ which is concentrated within 0.05 *pc* from the core of M15. It is possible that such a core would not only be adequate to provide the necessary perturbation for the observed warp but also survive the tidal disruption by the central black hole.

3.2. Numerical Results

For illustrative purposes, we present a numerical model to show that the observed properties of the warped disk in NGC 4258 can be regulated by the perturbation on a marginally self–gravitating disk due to a low–mass companion on an inclined orbit. Since we include the secular effect of the companion in an orbit with finite inclination through the perturbing potential Ψ'_{ext}, we consider equation (34) with $\omega_z = 0$ (i.e. the left hand side is zero).

In our numerical calculations, we solve equation (46) by dividing the interval $[R_i, R]$ into a grid $(r_j)_{j=1\ldots n}$ using a constant spacing Δr in the radial direction and such that $r_1 = R_i + \Delta r/2$ and $r_n = R - \Delta r/2$. We approximate equation (34) at $r = r_j, j = 1 \ldots n$ as a system of n linear equations for the n quantities $g_j \equiv g(r_j), j = 1 \ldots n$ (typically $n =$ 200–2000 in our calculations). The fact that $g_j =$ constant is a solution gives a solubility condition analagous to (47) from which the precession frequency may be found. The perturbing potential can then be modified so as to include the effects of the Coriolis force. The specification $g_1 = 0$, corresponding to zero disk inclination at the inner boundary, then enables the solution to be completed.

For computational convenience, we normalize the units such that $M = 1$, $R = 1$ and $\Omega_K(R) = 1$. The observations of Miyoshi *et al.* (1995) indicate that $M_D(R)$ of the disk is less than $10^{-2}M$. Marginally self–gravitating disks are then such that the disk semi–thickness is less than one percent of the radius. For illustrative purposes we have chosen a model for which $\Sigma = \Sigma_0(R/r - 1)$, with the constant Σ_0 being chosen such that $M_D(R) = 1.5 \times 10^{-3}M, \alpha = 0.1$, and the maximum value of H/r, $(H/r)_{max}$, to be $\sim 10^{-3}$ (see discussion in § 1).

The equilibrium structure of the disk is calculated adopting a polytropic index 1.5. We plot the Toomre parameter $Q = \kappa c_s/(\pi G \Sigma)$, with c_s being evaluated in the midplane for our model disk with $R_i = 0.1R$ in Figure 1. The minimum value of Q is approximately unity making the disk marginally self–gravitating.

For the companion providing the perturbation, we adopt an orbital inclination with respect to the disk of $\delta = \pi/4$. Results for other values of δ can be obtained by noting that to a reasonable approximation, the precession frequency is proportional to $\cos\delta$ and g is proportional to $\sin 2\delta$. We also take the companion's mass to be such that $M_p = M_D(R)$ and the orbital radius to be $D = 1.5R$.

From equation (34), we note that g is essentially independent of a scaling which reduces M_D, M_p, and H by the same factor. This scaling relation holds because Q is unaltered. However, for large $\alpha > H/r$, the small amount of twist induced by the viscosity is reduced by the same factor. Other things being equal, g scales with M_p. Results were found to be insensitive to the form of the surface density taper at the outer boundary.

The induced precession frequency in our model is $\omega_p = -7.9 \times 10^{-5}(J/J_{orb})\Omega(R)$. We are able to reproduce a warp with a relatively large amplitude using a perturbing potential which is much smaller than that due to the central mass, justifying the assumption of a small perturbation. An approximate estimate for $|g|$ may be obtained by estimating the magnitudes of terms in (6) and (34) as

$$|g| \sim \frac{3M_pR^3}{8M_D(R)D^3}|\sin 2\delta|. \tag{53}$$

This amplitude can be interpreted as being the product of the companion induced precession frequency and the time required for a bending wave to propagate through the disk. When this product is small, there is good communication between the different parts of the disk, and the warp is small. In the thin, self–gravitating disks we have considered here, wave propagation is mainly regulated by self–gravity which acts to preserve the disk's intrinsic spin vector.

In Figure 2.a, we plot both the real part of g and the imaginary part of $-g$ as functions of r. These quantities represent the inclination of the disk as seen edge on, viewed looking down the y and x–axis, where $\varphi = \pi/2$ and 0 respectively. We have also calculated g with just the self–gravitating term in (35). As expected (see § 2.4.6), the real part of g so obtained is hardly distinguishable from what we get when pressure and viscosity with $\alpha = 0.1$ are included. For comparison, Figure 2.b shows the real part of g obtained including both self–gravity and pressure terms but with very small $\alpha = 10^{-5}$. When $\alpha = 0$ the solution looks almost exactly the same. The case $\alpha = 0.1$ has been displayed again on the same plot for comparison. We observe that when α is small, the response oscillates around that associated with the larger α. This is due to the fact that when both self–gravity and pressure are present, both a slow and a fast bending wave can propagate in the disk (Papaloizou & Lin 1995). When α is large, the pressure term in (35) is small, so that only the fast wave corresponding to the purely self–gravitating case can propagate. In Figure 2.b we see the superposition of the fast and slow waves, the former having a larger wavelength than the latter. We have checked that the slow wave disappears once α becomes larger than H/r. We note that the precession frequency is the same in all cases. Hereafter, unless otherwise specified, we shall refer to the case where self–gravity, pressure and viscosity with $\alpha = 0.1$ are included.

In Figure 3, we plot a three dimensional view of the warped disk. For clarity, the vertical scale has been magnified.

The observations indicate the presence of a warp corresponding to $|g|$ reaching values of about

a few tenths at the disk edge. The results presented here reproduce a warp of the required magnitude. Since the outer region of the disk is mainly regulated by the external perturbation, the functional dependence of g on r in the outer part of the disk is insensitive to the choice of R_i. But the total range of $|g|$ increases as R_i decreases. This dependence is due to the fact that *in our model* the wave crossing time across the disk decreases as R_i increases. In the limit of small R_i, it becomes difficult for the outer parts of the disk to "communicate" with the inner parts such that the two regions become essentially disconnected.

We now consider the twist (the azimuth of the maximum of the real part of g at a given radius as a function of radius). This azimuth is given by $2\pi - \eta(r)$, with $\eta(r)$ being the argument of g. Figure 4 shows $\eta(r)$ as a function of r.

The total variation in η is about a few degrees and most of this variation occurs in the inner regions. A positive gradient $d\eta(r)/dr$ implies a negative gradient in the azimuth and a regression of the line of nodes which corresponds to a trailing twist. This trailing twist is produced by the second term of (35) when viscosity is added (see § 2.4.6). In the outer part of the disk, the magnitude of this term is a factor H/r smaller than the other terms. If it is completely neglected, as noted above, $|g|$ is hardly changed, indicating that its *only* significant contribution is to produce the small twist. The twist becomes larger in the inner part of the disk because the perturbation amplitude of the external forcing is weakened relative to the effect of inertial response and viscous stress which are contained in the second term of (35). The value of η at small r would then be smaller if Q were not as large there as it is in our model.

Accurate determination of the twist can provide useful data on the magnitude of α. According to the results in Figure 4, the magnitude of $rd\eta/dr \sim 1^{\circ}$ in the outer part of our model disk (with $\alpha = 0.1$) and an order of magnitude larger in the inner part. In the analysis of NGC 4258, Herrnstein *et al.* (1996) constructed models to fit the observed position and velocity of the maser sources. Some of their models include variations, along the radial direction, in both the inclination and the projected position angles of the disk's spin axis. For example, their model 3 is consistent with a small trailing twist in which $|rd\eta/dr| \sim 2^{\circ}$, although models without any twist fit the data equally well. We note that the warp amplitude is on the order of a tenth of the radius throughout this particular model.

4. Discussion

We have derived the tilt equation governing the inclination of a near Keplerian disk subject to global warping ($m = 1$) perturbations taking into account self–gravity and pressure. We have considered the case where the perturbation is due to a companion on an inclined orbit, but the analysis we have presented in this paper is valid for any $m = 1$ perturbing potential proportional to the coordinate z along the disk rotation axis. All effects that can cause evolution on the fastest possible timescale on the order of $H\Omega r^{-1}$ have been included. We also considered the effects of

viscosity, showing that pressure effects diminish in importance relative to those due to self–gravity once the viscosity parameter $\alpha > H/r$. The disk response and precession frequency have been derived using two different approaches that we have shown to be equivalent in the small inclination limit. We considered i) the excitation of small amplitude low–frequency bending modes about the orbital plane of the companion. Here, the time averaged potential of the companion is included in the equilibrium state, ii) the zero frequency response of a disk forced by the secular potential due to a companion in an orbit with finite inclination. It was found that when the precession frequency is small compared to the rate of wave propagation across the disk, the disk can undergo quasi rigid body precession with a small warp. Then the timescale for the inclination to decrease is found to be long compared to other timescales in the problem.

For an illustrative application, we have explored the possibility that the recently observed warped disk in the active galaxy NGC 4258 is produced by a binary companion. Our results indicate that it can be produced by a companion with a comparable mass to that of the observed disk. For a disk mass of $10^{-3}M = 4 \times 10^4~M_\odot$, the required mass for the companion is comparable to that contained in the cores of the densest globular clusters. Either these cores or black holes with comparable masses can survive the tidal disruption of the central black hole. The warp remains of modest magnitude as long as the companion induced precession time is everywhere significantly longer than the time required for a density wave to propagate through the disk.

If the disk and companion's mass are comparable, their tidal interaction would lead to the tidal truncation of the outer disk edge and the companion's orbital evolution on the viscous timescale of the disk. The exchange of angular momentum between the companion and the disk occurs through the excitation of density waves in the disk. For the parameters we adopted, the viscous evolution timescale of the disk is $\sim 8 \times 10^8~yr$ giving a significant lifetime in the current state.

Subject to the effect of dynamical friction, the companion may also undergo orbital decay as it interacts with the field stars in the nucleus of NGC 4258. For illustrative purposes, we adopt a model in which the density of the galactic core varies like r^{-n}, r being the distance from the galactic centre. Assuming that the velocity of the companion stays Keplerian as it spirals down to the centre, the timescale t_d on which the angular momentum L of the companion changes is then (Binney & Tremaine 1987)

$$t_d = \frac{L}{dL/dt} \sim \frac{C}{4\pi} \frac{1}{\omega} \frac{M}{M_p} \frac{M}{\rho_0 D^3}$$

where ρ_0 is the stellar density at the distance $r = D$ from the centre. C is a number which depends on various parameters of the galactic core (see Binney & Tremaine 1987 for more details). It is typically on the order of unity, and we then set $C = 1$. In our model, $D \sim 0.3~pc$. If we take for ρ_0 the value estimated for the galaxy M32, which has been resolved down to subparsec scale, we have ρ_0 between 10^6 (Lauer *et al.* 1992) and $10^7~M_\odot/pc^3$. This gives t_d between 10^7 and $10^8~yr$, and thus the companion loses angular momentum because of dynamical friction faster than it gains it

by interaction with the disk. For t_d to be at least on the order of the viscous timescale of the disk, we need $\rho_0 < 1.5 \times 10^5 \ M_\odot/pc^3$. This is not unreasonable, since we might expect the stars in the centre of NGC 4258 to be cleared out by the presence of the companion.

The fact that the companion spirals in with a lifetime comparable or smaller than the viscous timescale of the disk provides an additional argument why we do not expect the disk to have settled to equilibrium with zero inclination to the orbital plane (see discussion in § 1).

The authors wish to thank L.J. Greenhill for useful conversations. This work is supported by PPARC through grant GR/H/09454 and by NASA through grant NAG 53059. C.T. acknowledges support for this work by the Center for Star Formation Studies at NASA/Ames Research Center and the University of California at Berkeley and Santa–Cruz, and by the EU.

A. APPENDIX

We here show that the magnitude of the combination of the two terms in equation (25) given by

$$\int_{-\infty}^{\infty} \rho f_h' dz = \int_{-\infty}^{\infty} \left(\rho \, \frac{\partial \Psi_{Gh}'}{\partial z} + \rho_h' \, \frac{\partial \Psi_G}{\partial z} \right) dz, \tag{A1}$$

for global perturbations under conditions for which the Toomre parameter $Q \sim 1$, is of order $\Omega_K^2 \Sigma H^2 \xi_z / r^2$ for characteristic values of H and r and so can be neglected in our ordering scheme. Of course, because these terms derive from the disk self–gravity, for $Q \gg 1$, they are even less significant. We suppose that the disk material is all contained within $-H < z < H$, there being vaccuum outside this region. Here we and below we find it convenient to take H to be the maximum value of the semi–thickness of the disk.

We begin by finding Ψ_{Gh}'. We remark that interior to the disk Ψ_{Gh}' satisfies the Poisson equation

$$\nabla^2 \Psi_{Gh}' = 4\pi G \rho_h', \tag{A2}$$

while external to the disk it satisfies Laplace's equation. For the warping perturbations we consider, ρ_h' and Ψ_{Gh}' are odd functions of z. Thus $\partial \Psi_{Gh}' / \partial z$ is an even function of z and accordingly it has the same values on the top and bottom of the disk.

To solve (A2), we use the standard method of Hankel transforms (see, e.g., Binney & Tremaine 1987). That is we write (after removing the factors $\exp(i\varphi + \sigma t)$):

$$\Psi_{Gh}' \quad = \quad \int_0^{\infty} \Psi_k'(k, z) J_m(kr) \, k \, dk, \tag{A3}$$

$$\rho'_h = \int_0^\infty \rho'_k(k,z) J_m(kr)\; k\; dk, \tag{A4}$$

where $\rho'_k(k,z)$ and $\Psi'_k(k,z)$ are the Hankel transforms of $\rho'_h(r,z)$ and $\Psi'_{Gh}(r,z)$ respectively. The Bessel function is denoted by J_m in standard notation. Here $m=1$. For the global disturbances we are interested in, the significant values of $k \sim r^{-1}$ at some characteristic radius.

The Poisson equation (A2) gives

$$\frac{d^2\Psi'_k}{dz^2} - k^2\Psi'_k = 4\pi G\rho'_k. \tag{A5}$$

Above the disk, where there is no source ($\rho'_k = 0$), the solution is $\Psi'_k(k,z) = C(k)\exp(-kz)$, where $C(k)$ is an arbitrary function which depends only on k. Because Ψ'_k is antisymmetric with respect to reflection in the midplane, below the disk ($z<0$) we have $\Psi'_k(k,z) = -C(k)\exp(kz)$.

To find $C(k)$, we multiply (A5) by z and integrate through the thin disk to obtain

$$2\left(H\frac{d\Psi'_k}{dz} - \Psi'_k\right)_+ - k^2\int_{-H}^{H} z\Psi'_k dz = 4\pi G\int_{-\infty}^{\infty} z\rho'_k dz, \tag{A6}$$

where the subscript $+$ denotes evaluation in the vaccuum at the upper boundary of the disk where $z=H$.

From (A6) we find to within a multiplicative error of at most of order $|kH|$, this being a small quantity for global perturbations:

$$C(k) = -2\pi G\int_{-\infty}^{\infty} z\rho'_k dz. \tag{A7}$$

From this it similarly follows that at $z=H$,

$$\left(\frac{\partial\Psi'_k}{\partial z}\right)_+ = 2\pi Gk\int_{-\infty}^{\infty} z\rho'_k dz. \tag{A8}$$

From equation (A3), it similarly follows that at $z=H$,

$$\left(\frac{\partial\Psi'_{Gh}}{\partial z}\right)_+ = 2\pi G\int_{-\infty}^{\infty}\int_0^\infty z\rho'_k J_m(kr)\; k^2\; dk\; dz. \tag{A9}$$

We recall that for global density perturbations we expect that ρ'_k is significant only for $k \sim r^{-1}$ at a characteristic radius.

Returning to the expression of interest given by equation (A1), we first use Poisson's equation to express densities in terms of potentials:

$$\rho = \frac{\nabla^2 \Psi_G}{4\pi G}, \quad \rho'_h = \frac{\nabla^2 \Psi'_{Gh}}{4\pi G}. \tag{A10}$$

However, we shall retain only the derinatives with respect to z in the Poisson operators which leads to a multiplicative error of order H^2/r^2 for globally varying functions. This in turn leads to a multiplicative error of order H/r in the final expression we obtain so it may be neglected. Using (A10) in this way in (A1) we obtain

$$\int_{-\infty}^{\infty} \rho f'_h dz = \frac{1}{4\pi G} \int_{-H}^{H} \left(\frac{\partial^2 \Psi_G}{\partial z^2} \frac{\partial \Psi'_{Gh}}{\partial z} + \frac{\partial^2 \Psi'_{Gh}}{\partial z^2} \frac{\partial \Psi_G}{\partial z} \right) dz = \frac{1}{2\pi G} \left(\frac{\partial \Psi'_{Gh}}{\partial z} \frac{\partial \Psi_G}{\partial z} \right)_+, \tag{A11}$$

where an integration by parts has been performed and the symmetry properties of the potentials have been used.

We also note that for the equilibrium potential

$$\left(\frac{\partial \Psi_G}{\partial z} \right)_+ = 2\pi G \Sigma. \tag{A12}$$

Using this together with (A9) in (A11), we obtain

$$\int_{-\infty}^{\infty} \rho f'_h dz = 2\pi G \Sigma \int_{-\infty}^{\infty} \int_0^{\infty} z \rho'_k J_m(kr)\, k^2\, dk\, dz. \tag{A13}$$

Using (A4) this may be equivalently expressed as

$$\int_{-\infty}^{\infty} \rho f'_h dz = 2\pi G \Sigma \int_{-\infty}^{\infty} z \rho'_h dz \left[\frac{\int_{-\infty}^{\infty} \int_0^{\infty} z \rho'_k J_m(kr)\, k^2\, dk\, dz}{\int_{-\infty}^{\infty} \int_0^{\infty} z \rho'_k J_m(kr)\, k\, dk\, dz} \right]. \tag{A14}$$

We remark that the expression in square brackets gives a mean value of k which for global perturbations may be taken $\sim r^{-1}$ at some characteristic radius. Further from the expression (31) we can simply estimate that for global perturbations

$$\int_{-\infty}^{\infty} z \rho'_h dz \sim \frac{\Sigma H \xi_z}{r}. \tag{A15}$$

These taken together with the condition $Q \sim 1$, yields the required estimate

$$\int_{-\infty}^{\infty} \rho f'_h dz \sim \Omega_K^2 \Sigma H^2 \xi_z / r^2. \tag{A16}$$

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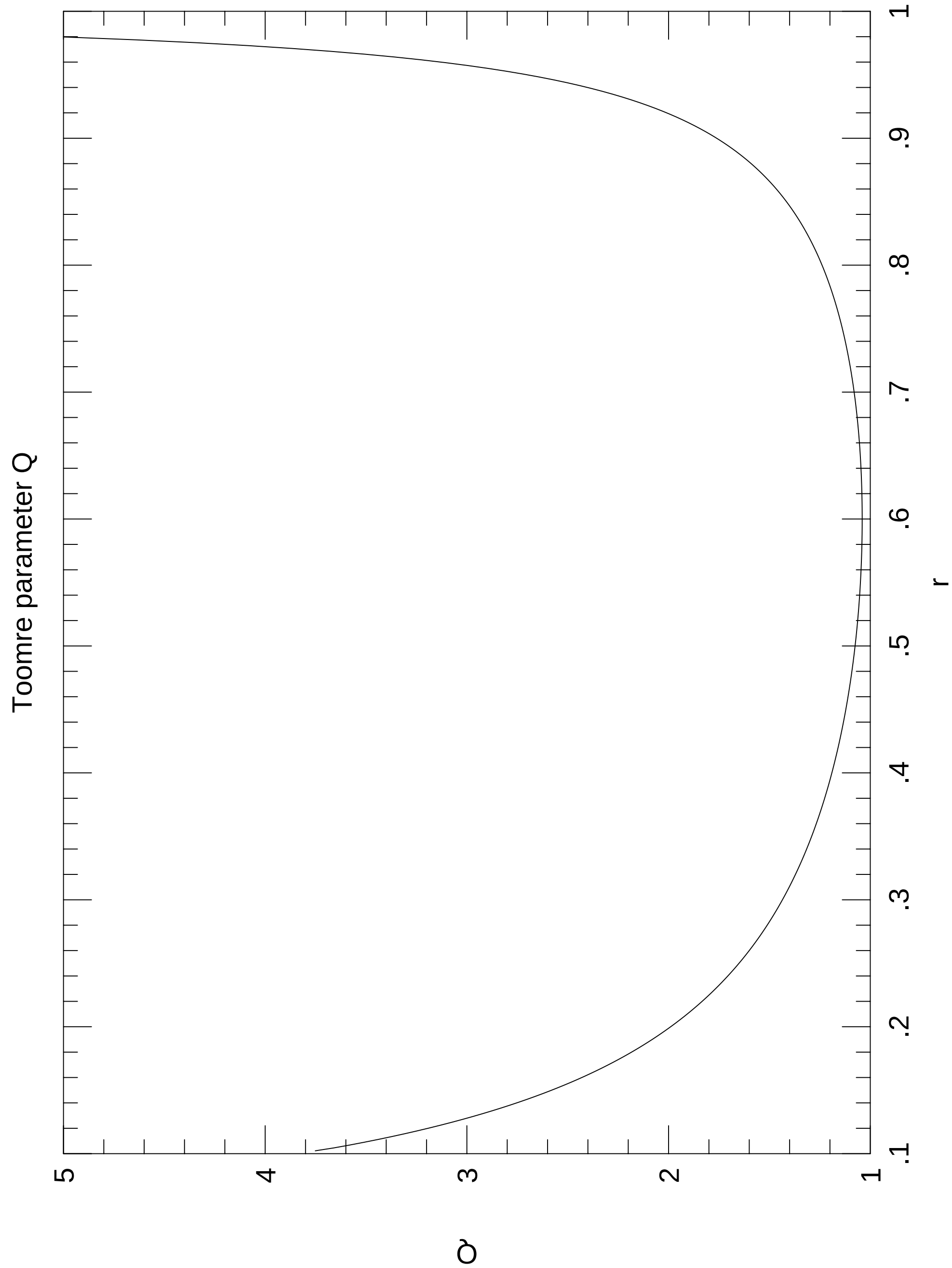

Fig. 1.— Toomre parameter $Q = \kappa c/(\pi G \Sigma)$ for $M_D(R) = 1.5.10^{-3} M$ and $(H/r)_{max} = 1.5.10^{-3}$. The other parameters are given in the text.

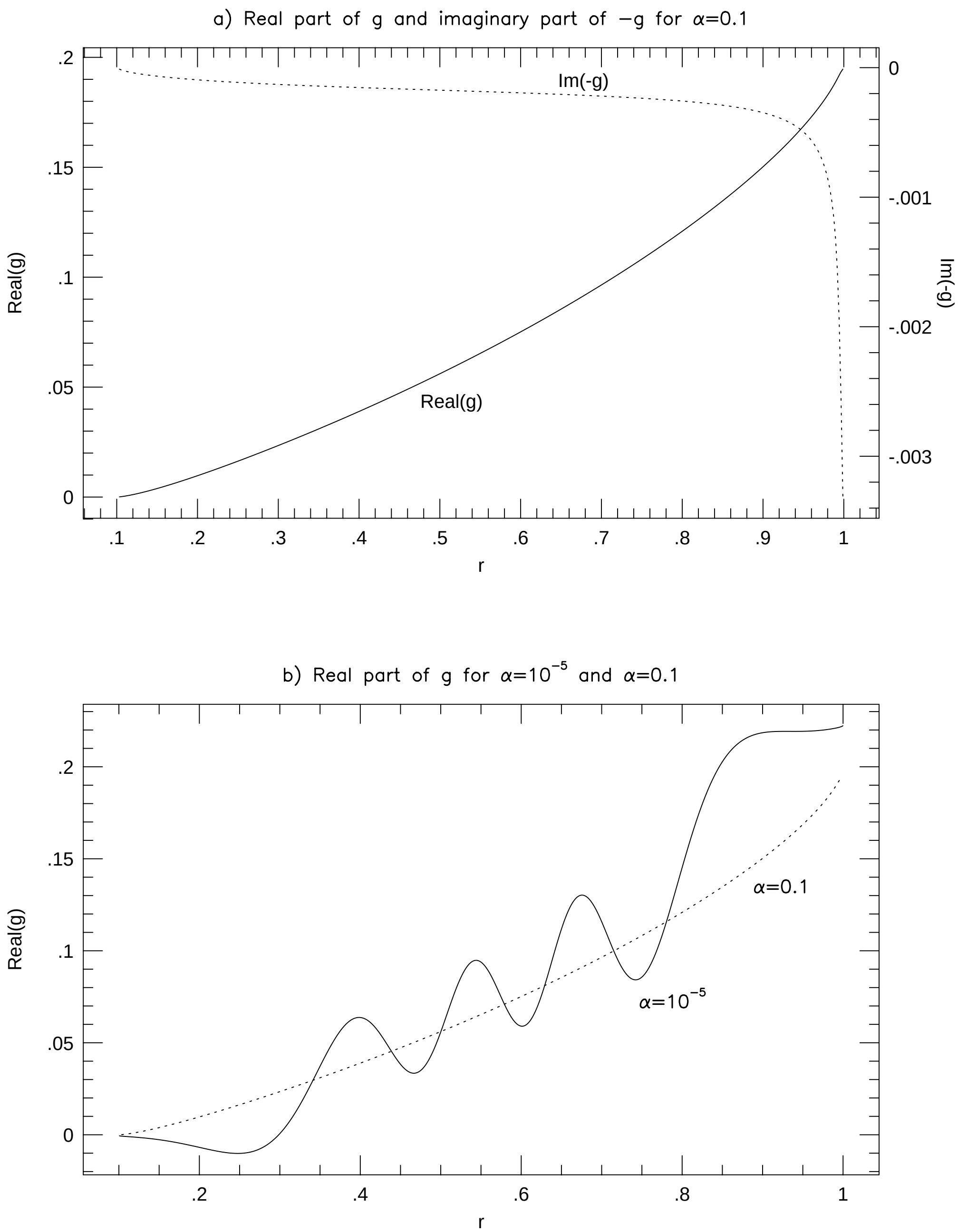

Fig. 2.— a) Real and imaginary parts of g and $-g$ respectively for a self–gravitating viscous disk with $\alpha = 0.1$. The other parameters are given in the text. b) Real part of g for the same disk but $\alpha = 10^{-5}$ and $\alpha = 0.1$.

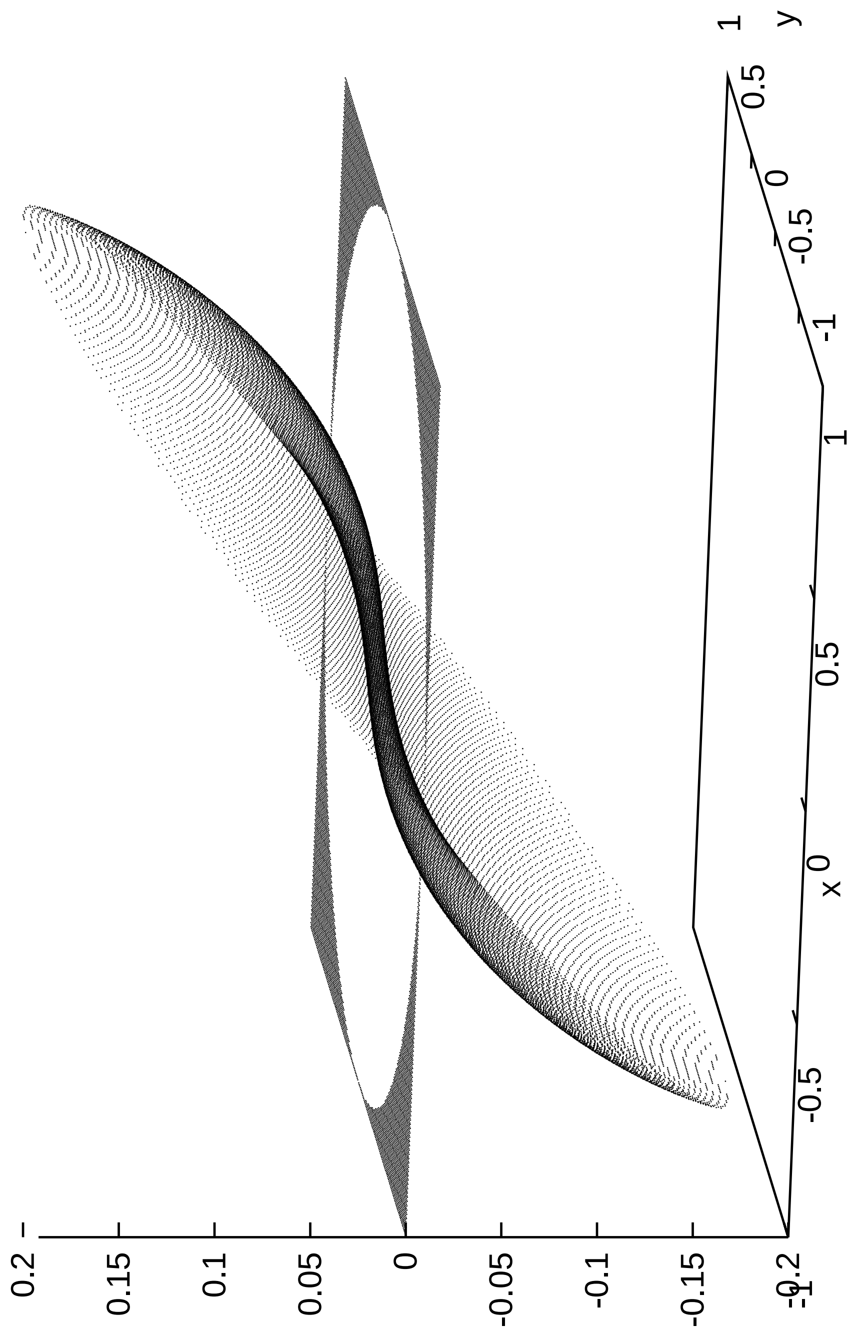

Fig. 3.— 3D view of the warped disk, with parameters given in the text. The vertical scale has been magnified. The plane $z = 0$ represents the plane of the undistorted disk.

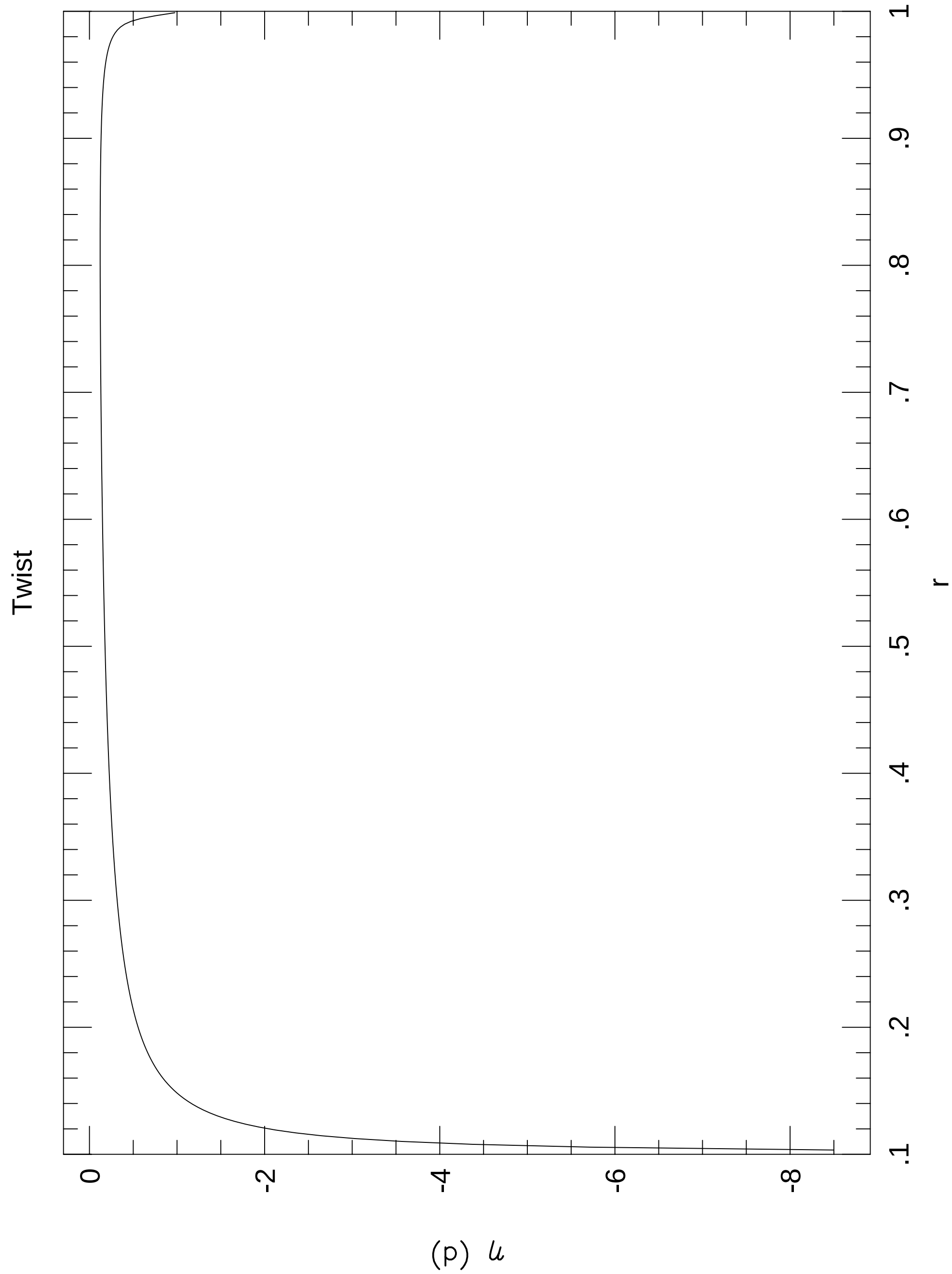

Fig. 4.— The phase angle $\eta(r)$, the variation of which defines the degree of twist, plotted in degrees as a function of r.